

Mesoscopic formulas of linear and angular momentum fluxes

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Cosserat theory [E. Cosserat and F. Cosserat, *Théorie des corps déformables*” (Herman,1909)] is a way to extend a simple Newtonian fluid model or bulk elasticity models to incorporate the underlying mesoscopic structures into the macroscopic theory. However, this theory, which contains asymmetric stress tensor and “couple stress tensor”, has been lacking its bottom-up theoretical basis which may correspond to the virial stress formula of Irving-Kirkwood theory of the gas hydrodynamics. Based on the momentum and angular momentum conservation laws, we will present a bottom-up formulation of the Cosserat-type theory in the way adapted to the dense cellular media. Mesoscopic formulas for the stress and couple stress tensors are given, which is reminiscent of the virial stress but without any assumptions about the interactions at the cell-cell interfaces.

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I. INTRODUCTION

In the most common continuum descriptions of deformable media, i.e. the hydrodynamics and elasticity, the internal structure of the media is not visible at the macroscopic level. However, the real materials, either passive or active, often have a number of important length scales. For real materials with intrinsic length scales, the average displacements only are not enough to describe the local deformations. Moreover, the linear macroscopic momentum flux, or the (minus of) stress field, is not convenient to describe all the local mechanical states. For example, a dense aggregate of living cells (e.g. *Dictyostelium Discoideum* in its migration stage [1]) involves the interaction among the cells through extended cell-cell interfaces.

In this context Cosserat and Cosserat [2] proposed a framework in 1909, while their framework was made known only some 50 years later to the general continuum mechanics community thanks to Günther [3]. See [4] and [5] for the historical reviews. In very brief the Cosserat theory replaces the single elasticity-hydrodynamics equation, $\partial_j \sigma_{ij} = 0$, for the symmetric total stress tensor σ_{ij} , by the two balance equations, $\partial_j \sigma_{ij} = 0$ and $\partial_j \mu_{ij} = -e_{ijk} \sigma_{jk}$, for the non-symmetric total stress tensor, σ_{ij} , and the so-called “couple stress tensor”, μ_{ij} , where ∂_j implies the spatial derivative with respect to the j th Cartesian component and e_{ijk} is the component of the Levi-Civita pseudo-tensor. Throughout this paper we suppose to be in the slow dynamic regime where the inertia effects are negligible

With all these success at least on the practical level, however, this theory has lacked the bottom-up construction, or the recipe of calculating the stress and couple stress tensors. The Cosserat theory has, therefore, remained a phenomenology. It will be very helpful if we

have a general framework to connect any microscopic model of structured medium to the macroscopic Cosserat theory. For comparison, Irving-Kirkwood theory [6] introduced the molecular-level expression of the stress for the gas, which is often called the virial stress. The equilibrium pressure driven using the virial theorem (see [7]p.130 or [8]p.190-191) and the virial stress has been shown to be consistent [9].

In the present paper, we will give a basis for the Cosserat theory by the bottom-up approach which is adapted for the cellular media. In our framework (i) the asymmetric stress is expressed in the form of virial stress, (ii) the couple stress is also expressed in a form very similar to the virial stress. While the details will be given below in the main part, the basic virial-like form of the stress is an appropriate average of (distance vector) \times (force vector) for the stress. For the couple stress the force above is replaced by the torque. In contrast with the molecular theories, however, the interactions among the cellular elements need not be of pairwise. In fact the dense cellular structures can contain highly complicated passive and active interactions through the interfaces among the cells. Our formalism enables to connect between the modeling analysis at the microscopic level and the macroscopic Cosserat framework. In the present paper we focus on the formalism: An application will be discussed elsewhere.

The key idea of our formalism is to look at the flow of momentum and angular momentum at the cell-cell interfaces. Whatsoever complicated are the origin and the behavior of the forces and torques among the cellular structures, our point is only to relate them to the macroscopic stress and couple stress. On the Cosserat theory a persistent doubt has been cast [5] concerning the identifiability of the presupposed element called “oriented rigid particles.” Also the conventional “derivation” of the theory using the virtual work of d’Alembert made less visible the principal roles of momentum and angular momentum conservation. Our formalism based only on the conservation laws will shed a new light not only on the Cosserat

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theory but also on the existing kinetic theory of fluids.

The organization of the paper is as follows. In the next section we first define the momentum and angular momentum conservation laws in terms of microscopic linear momentum flux $\vec{\mathcal{G}}$. Then we rewrite the microscopic balance equations into its mesoscopic version, or at the scale of cells, by introducing the forces and torques (i.e. momentum and angular momentum flows), $\vec{F}_{i,j}$ and $\vec{M}_{i,j}$, respectively, through the cell-cell interfaces. Then in § III we introduce what we call the *neighbor distribution function* $\hat{\rho}_{2FM}$ defined in (11). This function allows us rewrite the discrete mesoscopic balance equations with $\vec{F}_{i,j}$ and $\vec{M}_{i,j}$ in an apparently continuum form with keeping all the informations of the former. The continuum form is advantageous for the coarse-graining procedure. Moreover, this new function $\hat{\rho}_{2FM}$ bears a special redundancy property, which will also a crucial role later. In § IV we apply the spatial coarse-graining procedure to the mesoscopic balance equations expressed in terms of $\hat{\rho}_{2FM}$. Using the aforementioned redundancy of the latter, which survives after the coarse-graining, we finally reach the results which (i) reproduce Cosserat's balance equations in terms of the coarse-grained momentum and angular momentum fluxes, $\vec{\mathcal{G}}$ and $\vec{\mathcal{C}}$, respectively on the one hand, and (ii) give physically appealing formulas for these fluxes on the other hand. The results, which are given in (22) and (25) with the definition about the averaging (23). The last section § V is for the concluding discussion. Since we will highlight on the role of the momentum fluxes, we shall use the notations somehow different from the conventional ones. However, correspondences are simple and unique; $\vec{\sigma} = \{\sigma_{ij}\} \mapsto -\vec{\mathcal{G}}$ and $\vec{\mu} = \{\mu_{ij}\} \mapsto -\vec{\mathcal{C}}$.

II. MICROSCOPIC LEVEL AND CELL-LEVEL EXPRESSIONS OF THE BALANCE RELATIONS

A. Medium

We first make precise what we call “cellular” structure. We are interested in the three-dimensional packing of closed compartments, which we call “cells.” We distinguish each cell by an index, i . We assume that there is no interior free surfaces of these cells, being different from packed granular media. The size and shape of the cell can be distributed around typical ones. We denote by \vec{r}_i the center of volume of the i th cell. We also denote by Ω_i the space occupied by the i -th cell. The border of this volume, $\partial\Omega_i$, needs not to be of polyhedral shapes, and the number of the immediate neighbor cells for each cell need not to be the same for all the cells.

To any spatial domain Ω , either large or small, we can associate its “closure” domain, which we denote by $\tilde{\Omega}$, as the union of Ω_i of those cells whose center belong to Ω .

In equation, it reads

$$\tilde{\Omega} \equiv \bigcup_{\vec{r}_i \in \Omega} \Omega_i,$$

or schematically it looks like Fig. 1.

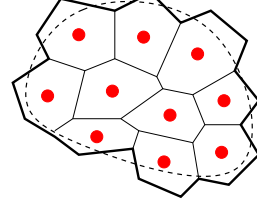


FIG. 1. Schematic definition of $\tilde{\Omega}$ (bounded by thick lines) for a given domain Ω (bounded by dashed curve). Cells are represented by 2D polygons and their centers inside Ω are marked by the thick dots. Thin lines are the “internal” cell-cell interfaces which do not belong to the border of $\tilde{\Omega}$. In reality the cells are three-dimensional and the cell-cell interfaces may not be flat.

B. Microscopic description of the momentum balances

We denote by $\vec{\mathcal{G}}$ the microscopic momentum flux tensor. It is defined such that $\vec{\mathcal{G}} \cdot d\vec{A}(\vec{r})$ gives the flow rate of momentum across the oriented infinitesimal surface element, $d\vec{A}(\vec{r})$, at the position \vec{r} . We ignore completely the “convective” part of the linear or angular momenta which is carried by the inertia or moment of inertia, respectively. Our basic starting point is the conservation laws of momentum and angular momentum,

$$\nabla \cdot \vec{\mathcal{G}}^t = 0, \quad 0 = \mathbf{e} : \vec{\mathcal{G}}^t,$$

where t means to take the transposition and \mathbf{e} is the Levi-Civita pseudo-tensor and the product “:” is such that $(\mathbf{e} : \vec{A})_\alpha = \sum_\beta \sum_\gamma \mathbf{e}_{\alpha\beta\gamma} A_{\beta\gamma}$ in the cartesian component representation. The second equation is nothing but the symmetry, $\vec{\mathcal{G}} = \vec{\mathcal{G}}^t$, but we wrote above in the form similar to the final Cosserat form summarized in the final section. In either form, it means that the description by $\vec{\mathcal{G}}$ is enough detailed that the balance of angular momentum can be expressed by the linear momentum flux alone [10]. From these setup we can derive the following statement:

Quasi-static conservation laws in terms of \mathcal{G} for a cellular element: About an individual cellular element, Ω_i , the balance of linear and angular momenta is represented by the relations:

$$\int_{\vec{r} \in \partial\Omega_i} \vec{\mathcal{G}} \cdot d\vec{A}(\vec{r}) = 0 \quad (1)$$

$$\int_{\vec{r} \in \partial\Omega_i} \vec{r} \wedge \overleftrightarrow{\mathcal{G}} \cdot d\vec{A}(\vec{r}) = 0, \quad (2)$$

where integral is done over the whole boundary $\partial\Omega_i$ of the i -th cell occupying the volume Ω_i , and $d\vec{A}(\vec{r})$ is the outward area element at the position $\vec{r} (\in \partial\Omega_i)$.

To derive (2) we can use the Gauss' theorem and then use both $\nabla \cdot \overleftrightarrow{\mathcal{G}}^t = 0$ and $\mathbf{e} : \overleftrightarrow{\mathcal{G}}^t = 0$.

C. Cell-level description of momentum balances

We then rewrite the above conservation laws in a form specific to the cellular medium. In other words we move from the space of \vec{r} to the space of the indices of the cells, $\{i\}$. This step allows us to reflect the characteristics of the cellular elements in the final coarse-grained description.

Definition of inter-cellular force $F_{i,j}$ and inter-cellular torque $\vec{M}_{i,j}$ across the cell-boundary: When the i th and j th cells share an interface, $\partial\Omega_i \cap \partial\Omega_j (\neq \emptyset)$, then the force and torque that the i th cell applies to the j th cell through this interface, which we denote by $\vec{F}_{i,j}$ and $\vec{M}_{i,j}$, respectively, are given by

$$\vec{F}_{i,j} \equiv \int_{\vec{r} \in \partial\Omega_i \cap \partial\Omega_j} \overleftrightarrow{\mathcal{G}} \cdot d\vec{A}_{i \rightarrow j}(\vec{r}) \quad (3)$$

$$\vec{M}_{i,j} \equiv \int_{\vec{r} \in \partial\Omega_i \cap \partial\Omega_j} (\vec{r} - \vec{r}_i) \wedge \overleftrightarrow{\mathcal{G}} \cdot d\vec{A}_{i \rightarrow j}(\vec{r}) - \vec{a}_{i,j} \wedge \vec{F}_{i,j}, \quad (4)$$

$$\vec{a}_{i,j} = \frac{\int_{\partial\Omega_i \cap \partial\Omega_j} (\vec{r} - \vec{r}_i) dA_{i \rightarrow j}(\vec{r})}{\int_{\partial\Omega_i \cap \partial\Omega_j} dA_{i \rightarrow j}(\vec{r})}, \quad (5)$$

where, the surface integral is done over the interface, $\partial\Omega_i \cap \partial\Omega_j$, with $d\vec{A}_{i \rightarrow j}(\vec{r})$ being the area element at \vec{r} , oriented from the i th cell toward the j th cell. The vector $\vec{a}_{i,j}$ is the relative position from \vec{r}_i to the areal center of the interface, $\partial\Omega_i \cap \partial\Omega_j$, see Fig. 2. In this figure the center-to-center vector $\vec{e}_{i,j}$ is also defined as

$$\vec{e}_{i,j} = \vec{r}_j - \vec{r}_i, \quad (6)$$

and we can see immediately the geometrical relation, $\vec{a}_{i,j} - \vec{a}_{j,i} = \vec{e}_{i,j}$. We can also see that the torque $\vec{M}_{i,j}$ in (4) is measured with respect to the center of the interface, $\partial\Omega_i \cap \partial\Omega_j$. While the apparent \vec{r}_i in (4) and that in (5) cancel with each other, we retain them since the physical meaning of the torque $\vec{M}_{i,j}$ is clearer in this form.

About these definitions, we notice redundancies in $\vec{F}_{i,j}$ and $\vec{M}_{i,j}$; the force that the i th cell receives at \vec{r}_i from the j th cell at $\vec{r}_j (= \vec{r}_i + \vec{e}_{i,j})$ is the minus of the force that the latter receives from the former. The similar statement holds also for the torque. Therefore, we have

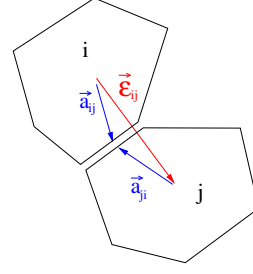


FIG. 2. Definition of $\vec{a}_{i,j}$ and $\vec{e}_{i,j}$. The common interface $\partial\Omega_i \cap \partial\Omega_j$ is denoted by \vec{S}_{ij} .

Reciprocity relations for $\vec{F}_{i,j}$ and $\vec{M}_{i,j}$: For the interface between the neighboring cell-pair, i and j , the force $F_{i,j}$ and torque $\vec{M}_{i,j}$ satisfy

$$\vec{F}_{i,j} + \vec{F}_{j,i} = \vec{0}, \quad \vec{M}_{i,j} + \vec{M}_{j,i} = \vec{0}. \quad (7)$$

Mathematically, the geometrical identity, $d\vec{A}_{i \rightarrow j}(\vec{r}) + d\vec{A}_{j \rightarrow i}(\vec{r}) = \vec{0}$ applied in (3) and (4) leads immediately to the above antisymmetric relations.

The definitions of $\vec{F}_{i,j}$ and $\vec{M}_{i,j}$ also allow to rewrite the microscopic continuum momentum balance equations (1) and (2) into the discretized form:

Kirchhoff-type laws for $\vec{F}_{i,j}$ and $\vec{M}_{i,j}$ about a cellular element : For the i th cellular element and its neighbors, the microscopic laws (1) and (2) implies the balance of the momentum and angular momentum flows:

$$\sum_j^{(i)} \vec{F}_{i,j} = \vec{0}, \quad (8)$$

$$\sum_j^{(i)} \left(\vec{M}_{i,j} + \vec{a}_{i,j} \wedge \vec{F}_{i,j} \right) = \vec{0}. \quad (9)$$

where the sum $\sum_j^{(i)}$ runs over all the cells indexed by j for which the i -th cell is immediate neighbor. To show these laws, we decompose the cell border $\partial\Omega_i$ in (1) and (2) using the identity, $\partial\Omega_i = \cup_j^{(i)} (\partial\Omega_i \cap \partial\Omega_j)$. Then the definitions (3) and (4) are used, respectively. As is evident from the definitions of $\vec{F}_{i,j}$ and $\vec{M}_{i,j}$ the apparent “pair-interaction” forms of (8) and (9) can admit any many-body interactions among the cellular elements.

Identities for the global conservation laws: Because of the flow conserving nature of the momentum and angular momentum, we can extend the surface integrals of a single cell that appeared on the left hand side (l.h.s.) of (1) and (2) to a packed composition of the cells. That is, for any “closure” surface $\hat{\Omega}$, there hold $\int_{\vec{r} \in \partial\hat{\Omega}} \overleftrightarrow{\mathcal{G}} \cdot d\vec{A}(\vec{r}) = \sum_{\vec{r}_i \in \Omega} \int_{\vec{r} \in \partial\Omega_i} \overleftrightarrow{\mathcal{G}} \cdot d\vec{A}(\vec{r})$ and $\int_{\vec{r} \in \partial\hat{\Omega}} \vec{r} \wedge \overleftrightarrow{\mathcal{G}} \cdot d\vec{A}(\vec{r}) = \sum_{\vec{r}_i \in \Omega} \int_{\vec{r} \in \partial\Omega_i} \vec{r} \wedge \overleftrightarrow{\mathcal{G}} \cdot d\vec{A}(\vec{r})$. Then, using as building blocks the flow elements $\vec{F}_{i,j}$ and $(\vec{M}_{i,j} + \vec{a}_{i,j} \wedge \vec{F}_{i,j})$

(see (3) and (4)), we have the following identities: For arbitrary “closure” surface $\hat{\Omega}$, there hold[11]

$$\begin{aligned} \int_{\vec{r} \in \partial \hat{\Omega}} \vec{\mathcal{G}} \cdot d\vec{A}(\vec{r}) &= \sum_{\vec{r}_i \in \Omega} \left\{ \sum_j^{(i)} \vec{F}_{i,j} \right\}, \\ \int_{\vec{r} \in \partial \hat{\Omega}} \vec{r} \wedge \vec{\mathcal{G}} \cdot d\vec{A}(\vec{r}) &= \sum_{\vec{r}_i \in \Omega} \left\{ \sum_j^{(i)} \left(\vec{M}_{i,j} + \vec{a}_{i,j} \wedge \vec{F}_{i,j} \right) \right\}. \end{aligned} \quad (10)$$

III. CELL-LEVEL LOCAL “NEIGHBOR DISTRIBUTION FUNCTION”

The next and crucial step is to rewrite (8) and (9) for a cell Ω_i in the forms which are more adapted to the

coarse-graining. The new idea, to our knowledge, is to use a local “neighbor distribution functions”, $\hat{\rho}_{2FM}$.

Definition of neighbor distribution function: At the cell-level, we will introduce an empirical and local simultaneous distribution function of the center distance \vec{e} , the cell-to-cell force \vec{F} and cell-to-cell torque \vec{M} associated to a cell at the position \vec{r} . (Throughout this paper, we will use the “hat” symbol like \hat{A} to denote those quantities which are *before* the coarse-graining.)

$$\hat{\rho}_{2FM}(\vec{e}, \vec{F}, \vec{M}, \vec{r}) \equiv \sum_i \sum_j^{(i)} \delta(\vec{e} - \vec{e}_{i,j}) \delta(\vec{F} - \vec{F}_{i,j}) \delta(\vec{M} - \vec{M}_{i,j}) \delta(\vec{r} - \vec{r}_i). \quad (11)$$

Such a simultaneous distribution function is not a technical tool but a necessary ingredient when we construct a bottom-up theory without assuming specific constitutive equations. Hereafter, for the simplicity of the notations, we will also use the “peripheral” or partially integrated neighbor distribution functions, $\hat{\rho}_{2F}(\vec{e}, \vec{F}, \vec{r}) \equiv \sum_i \sum_j^{(i)} \delta(\vec{F} - \vec{F}_{i,j}) \delta(\vec{r}_i - \vec{r}) \delta(\vec{r}_j - \vec{r}_i - \vec{e})$, and $\hat{\rho}_{2M}(\vec{e}, \vec{M}, \vec{r}) \equiv \sum_i \sum_j^{(i)} \delta(\vec{M} - \vec{M}_{i,j}) \delta(\vec{r}_i - \vec{r}) \delta(\vec{r}_j - \vec{r}_i - \vec{e})$. Also we will use the purely geometrical neighbor distribution, $\hat{\rho}_2(\vec{e}, \vec{r}) \equiv \sum_i \sum_j^{(i)} \delta(\vec{r}_i - \vec{r}) \delta(\vec{r}_j - \vec{r}_i - \vec{e})$. The further integration of $\hat{\rho}_2(\vec{e}, \vec{r})$ over \vec{e} yields $\int \hat{\rho}_2(\vec{e}, \vec{r}) d^3 \vec{e} = \sum_i \delta(\vec{r}_i - \vec{r}) \sum_j^{(i)} 1$, where $\sum_j^{(i)} 1$ is the number of neighbors of the i th cell. Finally the empirical (single) cell density function, $\hat{\rho}_1$, is defined by

$$\hat{\rho}_1(\vec{r}) \equiv \sum_i \delta(\vec{r}_i - \vec{r}). \quad (12)$$

As the neighbor density distribution, $\hat{\rho}_{2FM}$, contains the detailed informations about $\vec{e}_{i,j} = \vec{r}_j - \vec{r}_i$, $\vec{F}_{i,j}$ and $\vec{M}_{i,j}$, we will represent the integrals (10), which were once written using the sums $\sum_j^{(i)} \vec{F}_{i,j}$ and $\sum_j^{(i)} \vec{M}_{i,j}$, as the moment integrals of $\hat{\rho}_{2FM}$.

Global integrals using the neighbor distribution function: For arbitrary Ω ,

$$\int_{\vec{r} \in \partial \hat{\Omega}} \vec{\mathcal{G}} \cdot d\vec{A}(\vec{r}) = \int_{\vec{r} \in \Omega} \left[\iint \vec{F} \hat{\rho}_{2F}(\vec{e}, \vec{F}, \vec{r}) d^3 \vec{F} d^3 \vec{e} \right] d^3 \vec{r}, \quad (13)$$

$$\begin{aligned} \int_{\vec{r} \in \partial \hat{\Omega}} \vec{r} \wedge \vec{\mathcal{G}} \cdot d\vec{A}(\vec{r}) &= \int_{\vec{r} \in \Omega} \left[\iiint (\vec{M} + \frac{1}{2} \vec{e} \wedge \vec{F}) \hat{\rho}_{2FM}(\vec{e}, \vec{F}, \vec{M}, \vec{r}) d^3 \vec{e} d^3 \vec{F} d^3 \vec{M} \right] d^3 \vec{r}. \end{aligned} \quad (14)$$

The proofs of (13) and (14) are given in Appendix A, where the identities (10) are crucial.

Redundancy of $\hat{\rho}_{2FM}(\vec{e}, \vec{F}, \vec{M}, \vec{r})$: The utility of the distribution $\hat{\rho}_{2FM}$ is not only that it contains the whole local informations at the cell-level but also that this distribution inherits the reciprocity relations (7) in the form of redundancy.

$$\hat{\rho}_{2FM}(\vec{e}, \vec{F}, \vec{M}, \vec{r}) = \hat{\rho}_{2FM}(-\vec{e}, -\vec{F}, -\vec{M}, \vec{r} + \vec{e}). \quad (15)$$

The proof of (15) is given in Appendix B. The associated peripheral distributions should also obey

$$\hat{\rho}_{2F}(\vec{e}, \vec{F}, \vec{r}) = \hat{\rho}_{2F}(-\vec{e}, -\vec{F}, \vec{r} + \vec{e}). \quad (16)$$

$$\hat{\rho}_{2M}(\vec{e}, \vec{M}, \vec{r}) = \hat{\rho}_{2M}(-\vec{e}, -\vec{M}, \vec{r} + \vec{e}). \quad (17)$$

These redundancies give the relations among the moment integrals such as

$$\begin{aligned} \iint \vec{F} \hat{\rho}_{2F}(\vec{e}, \vec{F}, \vec{r}) d^3 \vec{F} d^3 \vec{e} &= \frac{1}{2} \iint \vec{F} \left[\hat{\rho}_{2F}(\vec{e}, \vec{F}, \vec{r}) - \hat{\rho}_{2F}(\vec{e}, \vec{F}, \vec{r} - \vec{e}) \right] d^3 \vec{F} d^3 \vec{e} \end{aligned} \quad (18)$$

$$\begin{aligned} \iint \vec{e} \otimes \vec{e} \otimes \vec{F} \hat{\rho}_{2F}(\vec{e}, \vec{F}, \vec{r}) d^3 \vec{F} d^3 \vec{e} &+ \iint \vec{e} \otimes \vec{e} \otimes \vec{F} \hat{\rho}_{2F}(\vec{e}, \vec{F}, \vec{r} - \vec{e}) d^3 \vec{F} d^3 \vec{e} = 0 \end{aligned} \quad (19)$$

The proofs of (18) and (19) are given in Appendix C. These relations are useful because the integrals, $\iint \vec{F} \hat{\rho}_{2F}(\vec{e}, \vec{F}, \vec{r}) d^3 \vec{F} d^3 \vec{e}$ and $\iint \vec{M} \hat{\rho}_{2M}(\vec{e}, \vec{M}, \vec{r}) d^3 \vec{M} d^3 \vec{e}$, are delicate ones containing the cancellation of the dominant order. For example, (16) means that the typical values of \vec{F} for a given \vec{e} and for $(-\vec{e})$ will be roughly opposite at a given

\vec{r} or $\vec{r} + \vec{\epsilon}$. The above relations render them into harmless forms. Correct treatment of such integral is a key point of the coarse-graining procedure (see below). The above redundancy should be maintained upon the coarse-graining procedure.

IV. COARSE-GRAINING OF THE CONSERVATION LAWS

A. Coarse-grained neighbor distributions

We now consider the situations where the *statistics* of $\{\vec{\epsilon}_{ij}\}$, $\{\vec{F}_{ij}\}$ and $\{\vec{M}_{ij}\}$ over the volume of any “fluid particle” occupying a macroscopically small but microscopically large spatial domain varies only slowly as function of the representative position \vec{r} of that fluid particle. Note that we do not require the near regularity of the cellular structure but suppose only the statistical near homogeneity of the microscopic state. If the above condition is fulfilled, one may replace the empirical neighbor distribution function, $\hat{\rho}_{2FM}(\vec{\epsilon}, \vec{F}, \vec{M}, \vec{r})$, by a continuous function, $\rho_{2FM}(\vec{\epsilon}, \vec{F}, \vec{M}, \vec{r})$, which we call the coarse-grained neighbor distribution function, as long as we use the latter for calculating the moment integrals such as those appeared in (18) and (19). By the above definition of coarse-graining, those the moment integrals calculated using ρ_{2FM} are slowly varying functions of \vec{r} . Their spatial derivative with respect to \vec{r} , which we denote by ∇ , must, therefore, satisfy $\|\vec{\epsilon} \cdot \nabla\| \ll 1$ when $\|\vec{\epsilon}\|$ is of the typical center-to-center distance of neighboring cell pairs.

From the coarse-grained neighbor distribution ρ_{2FM} we can induce the associated [peripheral] distributions functions, $\rho_{2F}(\vec{\epsilon}, \vec{F}, \vec{r})$, $\rho_{2M}(\vec{\epsilon}, \vec{M}, \vec{r})$, $\rho_2(\vec{\epsilon}, \vec{r})$ and $\rho_1(\vec{r})$, just as are done for the empirical distribution functions. In place of the empirical number of cell neighbors, $\sum_j^{(i)} 1$, we introduce the coarse-grained number of neighbors, $Z(\vec{r})$, defined through

$$Z(\vec{r})\rho_1(\vec{r}) \equiv \int \rho_2(\vec{\epsilon}, \vec{r}) d^3\vec{\epsilon}. \quad (20)$$

B. Coarse-grained conservation law

The redundancy relations (15), (16) and (17) for the empirical distribution functions $\hat{\rho}_*$ (* stands for $_{2FC}$ etc.) should be inherited by the coarse-grained ones ρ_* as the common identities of all the empirical ones. Using these redundancy relations, we analyze the moment integrals of ρ_{2FP} and reach the central results of the present paper:

Coarse-grained momentum and its balance law: The coarse-grained momentum flux \vec{G} that satisfies the local (coarse-grained) momentum conservation,

$$\nabla \cdot \vec{G} = \vec{0}. \quad (21)$$

is identified to be

$$\vec{G}(\vec{r}) \equiv Z(\vec{r})\rho_1(\vec{r})\langle \frac{\vec{\epsilon}}{2} \otimes \vec{F} \rangle_2(\vec{r}). \quad (22)$$

where $\langle \psi \rangle_2(\vec{r})$ stands for the average of ψ weighted by the coarse-grained neighbor distribution ρ_{2FM} ,

$$\langle \psi \rangle_2(\vec{r}) \equiv \frac{\iiint \psi \rho_{2FM}(\vec{\epsilon}, \vec{F}, \vec{M}, \vec{r}) d^3\vec{M} d^3\vec{F} d^3\vec{\epsilon}}{\iiint \rho_{2FM}(\vec{\epsilon}, \vec{F}, \vec{M}, \vec{r}) d^3\vec{M} d^3\vec{F} d^3\vec{\epsilon}}$$

$$= \frac{\iiint \psi \rho_{2FM}(\vec{\epsilon}, \vec{F}, \vec{M}, \vec{r}) d^3\vec{M} d^3\vec{F} d^3\vec{\epsilon}}{Z(\vec{r})\rho_1(\vec{r})}. \quad (23)$$

Also $\vec{X} \otimes \vec{Y}$ denotes the tensor whose ij component is $X_i Y_j$. The key of the proof is to use (18) or, more precisely, its coarse-grained version with $\hat{\rho}_{2F}$ being replaced by ρ_{2F} . The details are given in Appendix D.

Coarse-grained angular momentum and its balance law: The coarse-grained angular momentum flux \vec{C} that satisfies

$$\nabla \cdot \vec{C} = -\mathbf{e} : \vec{G}, \quad (24)$$

is identified to be

$$\vec{C}(\vec{r}) \equiv Z(\vec{r})\rho_1(\vec{r})\langle \frac{\vec{\epsilon}}{2} \otimes \vec{M} \rangle_2(\vec{r}). \quad (25)$$

The derivation is similar to the linear momentum case, see Appendix E. As mentioned in the Introduction the results are reminiscent of the virial stress formula [6, 9].

V. CONCLUSION AND DISCUSSION

We have found the mesoscopic formulas for the coarse-grained momentum flux \vec{G} and angular momentum flux \vec{C} :

$$\vec{G} \equiv Z\rho_1\langle \frac{\vec{\epsilon}}{2} \otimes \vec{F} \rangle_2, \quad \vec{C} \equiv Z\rho_1\langle \frac{\vec{\epsilon}}{2} \otimes \vec{M} \rangle_2.$$

These fluxes which satisfy the Cosserat-type balance equations:

$$\nabla \cdot \vec{G} = \vec{0}, \quad \nabla \cdot \vec{C} = -\mathbf{e} : \vec{G}.$$

The spatial resolution of the coarse grained fields \vec{G} and \vec{C} is limited by the typical magnitude of $\vec{\epsilon}$ that appears in the above mesoscopic formulas. We recall that the conventional Cosserat theory uses the stress $\vec{\sigma}$ and the couple stress $\vec{\mu}$, which are related to the fluxes \vec{G} and \vec{C} through the rules, $\vec{\sigma} = -\vec{G}$ and $\vec{\mu} = -\vec{C}$.

In order to apply to a specific setup, we need to supplement with proper boundary conditions adapted to the characteristic of the system. Unlike the conventional hydrodynamics of viscous fluids, the no-slip boundary condition is not a priori assured and we should carefully chose the appropriate ones case by case. The redundancy in the pair neighbor distribution ρ_{2FM} can be extended for the three or more cells that are neighboring with each other. The redundancy relations are then more complicated but richer in their implication. In practice when the forces \vec{F}_{ij} and torques \vec{M}_{ij} reflect the more-than two-body correlations such distributions are needed at least in the midst of the bottom-up calculations. A work is under way to use such distributions to model a dense aggregate of active cells, which will be discussed elsewhere [12]. The present framework gives a basis for modeling mesoscopic system, either active or passive. While the framework was developed for the cellular structure, it may be a challenge to reformulate the theory for the soft granular materials at high densities (see [13] that presents the bottom-up theory in a little different approach) or for the random elastic networks.

Appendix A: proof of (13) and (14)

First we show

$$\sum_{\vec{r}_i \in \Omega} \left\{ \sum_j^{(i)} \vec{F}_{i,j} \right\} = \int_{\vec{r} \in \Omega} \iint \vec{F} \hat{\rho}_{2F}(\vec{\epsilon}, \vec{F}, \vec{r}) d^3 \vec{F} d^3 \vec{\epsilon} d^3 \vec{r} \quad (\text{A1})$$

and

$$\sum_{\vec{r}_i \in \Omega} \left\{ \sum_j^{(i)} \vec{M}_{i,j} \right\} = \int_{\vec{r} \in \Omega} \iint \vec{M} \hat{\rho}_{2M}(\vec{\epsilon}, \vec{M}, \vec{r}) d^3 \vec{M} d^3 \vec{\epsilon} d^3 \vec{r} \quad (\text{A2})$$

for arbitrary Ω . Eq. (A1) holds because

$$\begin{aligned} \sum_{\vec{r}_i \in \Omega} \left\{ \sum_j^{(i)} \vec{F}_{i,j} \right\} &= \int_{\vec{r} \in \Omega} \int \sum_i \sum_j \vec{F}_{i,j} \delta(\vec{r}_j - \vec{r}) \delta(\vec{r}_j - \vec{r}_i - \vec{\epsilon}) d^3 \vec{\epsilon} d^3 \vec{r} \\ &= \int_{\vec{r} \in \Omega} \iint \sum_i \sum_j \vec{F} \delta(\vec{F} - \vec{F}_{i,j}) \delta(\vec{r}_j - \vec{r}) \delta(\vec{r}_j - \vec{r}_i - \vec{\epsilon}) d^3 \vec{F} d^3 \vec{\epsilon} d^3 \vec{r} \\ &= \int_{\vec{r} \in \Omega} \iint \vec{F} \hat{\rho}_{2F}(\vec{\epsilon}, \vec{F}, \vec{r}) d^3 \vec{F} d^3 \vec{\epsilon} d^3 \vec{r} \end{aligned}$$

The equality for the torque (A2) can be shown similarly. Next, combining the above results with (10) we have the identities,

$$\int_{\vec{r} \in \partial \tilde{\Omega}} \vec{r} \wedge \overleftrightarrow{\mathcal{G}} \cdot d\vec{A}(\vec{r}) = - \int_{\vec{r} \in \Omega} \iint \vec{M} \hat{\rho}_{2M}(\vec{\epsilon}, \vec{M}, \vec{r}) d^3 \vec{M} d^3 \vec{\epsilon} d^3 \vec{r} - \sum_{\vec{r}_i \in \Omega} \left\{ \sum_j^{(i)} \vec{a}_{i,j} \wedge \vec{F}_{i,j} \right\} \quad (\text{A3})$$

On the r.h.s., $\sum_{\vec{r}_i \in \Omega} \sum_j^{(i)} (\vec{a}_{i,j} \wedge \vec{F}_{i,j})$ can also be represented as a moment integral: we first notice that this sum is equal to $\frac{1}{2} \sum_{\vec{r}_i \in \Omega} \sum_j^{(i)} (\vec{a}_{i,j} - \vec{a}_{j,i}) \wedge \vec{F}_{i,j}$. Then, by the geometrical identity $\vec{a}_{i,j} - \vec{a}_{j,i} = \vec{\epsilon}_{i,j}$, we have $\sum_{\vec{r}_i \in \Omega} \sum_j^{(i)} (\vec{a}_{i,j} \wedge \vec{F}_{i,j}) = \frac{1}{2} \sum_{\vec{r}_i \in \Omega} \sum_j \vec{\epsilon}_{i,j} \wedge \vec{F}_{i,j}$, which is written as $\frac{1}{2} \int_{\vec{r} \in \Omega} \left(\iint \vec{\epsilon} \wedge \vec{F} \rho_{2F}(\vec{\epsilon}, \vec{M}, \vec{r}) d^3 \vec{\epsilon} d^3 \vec{M} \right) d^3 \vec{r}$. Therefore, (A3) reads (14). (*End of proof*)

Appendix B: Proof of (15)

In the definition of $\hat{\rho}_{2FM}$ (see (11))

$$\hat{\rho}_{2FM}(\vec{\epsilon}, \vec{F}, \vec{M}, \vec{r}) \equiv \sum_i \sum_j^{(i)} \delta(\vec{r}_j - \vec{r}_i - \vec{\epsilon}) \delta(\vec{F} - \vec{F}_{i,j}) \delta(\vec{M} - \vec{M}_{i,j}) \delta(\vec{r} - \vec{r}_i)$$

we change the argument as follows;

$$\begin{aligned} \hat{\rho}_{2FM}(-\vec{\epsilon}, -\vec{F}, -\vec{M}, \vec{r} + \vec{\epsilon}) &= \sum_i \sum_j^{(i)} \delta(\vec{r}_j - \vec{r}_i - \vec{\epsilon}) \delta(\vec{F} + \vec{F}_{i,j}) \delta(\vec{M} + \vec{M}_{i,j}) \delta(\vec{r} + \vec{\epsilon} - \vec{r}_i) \\ &= \sum_i \sum_j^{(i)} \delta(\vec{r}_j - \vec{r}_i - \vec{\epsilon}) \delta(\vec{F} + \vec{F}_{i,j}) \delta(\vec{M} + \vec{M}_{i,j}) \delta(\vec{r} - \vec{r}_j). \end{aligned} \quad (\text{B1})$$

Using the fact that the counting of all the immediate neighbors through the sum, $\sum_i \sum_j^{(i)}$, is equivalent to that through $\sum_j \sum_i^{(j)}$, the last line becomes

$$\hat{\rho}_{2FM}(-\vec{\epsilon}, -\vec{F}, -\vec{M}, \vec{r} + \vec{\epsilon}) = \sum_j \sum_i^{(j)} \delta(\vec{r}_j - \vec{r}_i - \vec{\epsilon}) \delta(\vec{F} + \vec{F}_{i,j}) \delta(\vec{M} + \vec{M}_{i,j}) \delta(\vec{r} - \vec{r}_j)$$

Finally, the reciprocity relations, $\vec{F}_{i,j} = -\vec{F}_{j,i}$ and $\vec{M}_{i,j} = -\vec{M}_{j,i}$ gives

$$\begin{aligned} \hat{\rho}_{2FM}(-\vec{\epsilon}, -\vec{F}, -\vec{M}, \vec{r} + \vec{\epsilon}) &= \sum_j \sum_i^{(j)} \delta(\vec{r}_j - \vec{r}_i - \vec{\epsilon}) \delta(\vec{F} - \vec{F}_{j,i}) \delta(\vec{M} - \vec{M}_{j,i}) \delta(\vec{r} - \vec{r}_j) \\ &= \hat{\rho}_{2FM}(\vec{\epsilon}, \vec{F}, \vec{M}, \vec{r}). \end{aligned} \quad (\text{B2})$$

We thus arrived at the basic redundancy relationship (15) claimed above. The associated relations for the peripheral distributions can be obtained by integrating over either \vec{M} or \vec{F} . (*End of proof.*)

Appendix C: Proof of (18) and (19)

For (18),

$$\begin{aligned} \iint \vec{F} \hat{\rho}_{2F}(\vec{\epsilon}, \vec{F}, \vec{r}) d^3 \vec{F} d^3 \vec{\epsilon} &= \frac{1}{2} \left[\iint \vec{F} \hat{\rho}_{2F}(\vec{\epsilon}, \vec{F}, \vec{r}) d^3 \vec{F} d^3 \vec{\epsilon} + \iint \vec{F} \hat{\rho}_{2F}(-\vec{\epsilon}, -\vec{F}, \vec{r} + \vec{\epsilon}) d^3 \vec{F} d^3 \vec{\epsilon} \right] \\ &= \frac{1}{2} \left[\iint \vec{F} \hat{\rho}_{2F}(\vec{\epsilon}, \vec{F}, \vec{r}) d^3 \vec{F} d^3 \vec{\epsilon} + \iint (-\vec{F}) \hat{\rho}_{2F}(\vec{\epsilon}, \vec{F}, \vec{r} + (-\vec{\epsilon})) d^3 \vec{F} d^3 \vec{\epsilon} \right] \\ &= \frac{1}{2} \iint \vec{F} [\hat{\rho}_{2F}(\vec{\epsilon}, \vec{F}, \vec{r}) - \hat{\rho}_{2F}(\vec{\epsilon}, \vec{F}, \vec{r} - \vec{\epsilon})] d^3 \vec{F} d^3 \vec{\epsilon}. \end{aligned} \quad (C1)$$

For (19)

$$\begin{aligned} \iint \vec{\epsilon} \otimes \vec{\epsilon} \otimes \vec{F} \hat{\rho}_{2F}(\vec{\epsilon}, \vec{F}, \vec{r}) d^3 \vec{F} d^3 \vec{\epsilon} &= \iint \vec{\epsilon} \otimes \vec{\epsilon} \otimes \vec{F} \hat{\rho}_{2F}(-\vec{\epsilon}, -\vec{F}, \vec{r} + \vec{\epsilon}) d^3 \vec{F} d^3 \vec{\epsilon} \\ &= - \iint \vec{\epsilon} \otimes \vec{\epsilon} \otimes \vec{F} \hat{\rho}_{2F}(\vec{\epsilon}, \vec{F}, \vec{r} - \vec{\epsilon}) d^3 \vec{F} d^3 \vec{\epsilon}. \end{aligned} \quad (C2)$$

By adding the l.h.s. and the 2nd line on the r.h.s., we have (19). (*End of proof*)

Appendix D: Proof of (21) and (22)

We look at (13) and rewrite its right hand side. Because ψ will not contain \vec{M} in this issue, we will use ρ_{2F} to simplify the notation. According to (18) or, more precisely, to its coarse-grained version,

$$\iint \vec{F} \rho_{2F}(\vec{\epsilon}, \vec{F}, \vec{r}) d^3 \vec{F} d^3 \vec{\epsilon} = \frac{1}{2} \iint \vec{F} [\rho_{2F}(\vec{\epsilon}, \vec{F}, \vec{r}) - \rho_{2F}(\vec{\epsilon}, \vec{F}, \vec{r} - \vec{\epsilon})] d^3 \vec{F} d^3 \vec{\epsilon}. \quad (D1)$$

The integral on the r.h.s. can be expanded using its slowly varying nature about \vec{r} :

$$\begin{aligned} (\text{r.h.s. of (D1)}) &= \frac{1}{2} \nabla \cdot \left\{ \iint \vec{\epsilon} \otimes \vec{F} \rho_{2F}(\vec{\epsilon}, \vec{F}, \vec{r}) d^3 \vec{F} d^3 \vec{\epsilon} \right\} \\ &\quad + \frac{1}{2} \nabla \nabla : \left\{ \iint \vec{\epsilon} \otimes \vec{\epsilon} \otimes \vec{F} \rho_{2F}(\vec{\epsilon}, \vec{F}, \vec{r}) d^3 \vec{F} d^3 \vec{\epsilon} \right\} + \mathcal{O}(\|\vec{\epsilon} \cdot \nabla\|^3). \end{aligned} \quad (D2)$$

Now the second term on the r.h.s., which is at most $\mathcal{O}(\|\vec{\epsilon} \cdot \nabla\|^2)$, is in fact only $\mathcal{O}(\|\vec{\epsilon} \cdot \nabla\|^3)$ because the $\vec{\epsilon} \cdot \nabla$ expansion of (19) shows that

$$2 \iint \vec{\epsilon} \otimes \vec{\epsilon} \otimes \vec{F} \hat{\rho}_{2F}(\vec{\epsilon}, \vec{F}, \vec{r}) d^3 \vec{F} d^3 \vec{\epsilon} - \nabla \cdot \iint \vec{\epsilon} \otimes \vec{\epsilon} \otimes \vec{F} \hat{\rho}_{2F}(\vec{\epsilon}, \vec{F}, \vec{r}) d^3 \vec{F} d^3 \vec{\epsilon} \simeq 0, \quad (D3)$$

therefore,

$$\begin{aligned} \nabla \nabla : \left[\iint \vec{\epsilon} \otimes \vec{\epsilon} \otimes \vec{F} \rho_{2F}(\vec{\epsilon}, \vec{F}, \vec{r}) d^3 \vec{F} d^3 \vec{\epsilon} \right] &\simeq \nabla \nabla : \left[\frac{1}{2} \nabla \cdot \left\{ \iint \vec{\epsilon} \otimes \vec{\epsilon} \otimes \vec{F} \rho_{2F}(\vec{\epsilon}, \vec{F}, \vec{r}) d^3 \vec{F} d^3 \vec{\epsilon} \right\} \right] \\ &= \mathcal{O}(\|\vec{\epsilon} \cdot \nabla\|^3). \end{aligned} \quad (D4)$$

Therefore, the first term on the r.h.s. of (D2) is a good approximation with an error of only $\mathcal{O}(\|\vec{\epsilon} \cdot \nabla\|^3)$. Now rewriting (D2) using the notation of $\langle \vec{\epsilon} \otimes \vec{F} \rangle_2$ and the definition (22), the equation (D1) becomes

$$\iint \vec{F} \rho_{2F}(\vec{\epsilon}, \vec{F}, \vec{r}) d^3 \vec{F} d^3 \vec{\epsilon} = \nabla \cdot \overset{\leftrightarrow}{G} + \mathcal{O}(\|\vec{\epsilon} \cdot \nabla\|^3).$$

The (13) takes the form: $\int_{\vec{r} \in \partial \Omega} \overset{\leftrightarrow}{G} \cdot d\vec{A}(\vec{r}) = \int_{\vec{r} \in \Omega} \nabla \cdot \overset{\leftrightarrow}{G} d^3 \vec{r}$. for arbitrary Ω up to the errors of $\mathcal{O}(\|\vec{\epsilon} \cdot \nabla\|^3)$. This justifies to identify $\overset{\leftrightarrow}{G}$ as the coarse-grained momentum flux. In terms of the suffix, it is rather the transpose, $\overset{\leftrightarrow t}{G}$, that corresponds to $\overset{\leftrightarrow}{G}$ due to our choice of representation of the latter. Finally (2) leads to (21). (*End of proof*)

Appendix E: Proof of (24) and (25)

By the completely parallel argument as the proof of the previous theorem, a part of the integral on the r.h.s. of (14) can be rewritten as:

$$\int_{\vec{r} \in \Omega} \iint \vec{M} \hat{\rho}_{2M}(\vec{\epsilon}, \vec{M}, \vec{r}) d^3 \vec{M} d^3 \vec{\epsilon} d^3 \vec{r} = \nabla \cdot \overset{\leftrightarrow}{C} + \mathcal{O}(\|\vec{\epsilon} \cdot \nabla\|^3), \quad (E1)$$

where we have used the definition (23) and (25). Besides, the remaining part of the integral in (14), i.e., $-\int_{\vec{r} \in \Omega} \left[\iint \frac{1}{2} \vec{\epsilon} \wedge \vec{F} \cdot \hat{\rho}_{2FM}(\vec{\epsilon}, \vec{F}, \vec{M}, \vec{r}) d^3 \vec{\epsilon} d^3 \vec{M} d^3 \vec{F} \right] d^3 \vec{r}$, can be expressed in terms of \vec{G} defined in (22):

$$\begin{aligned} \int_{\vec{r} \in \Omega} \left(\iint \frac{1}{2} \vec{\epsilon} \wedge \vec{F} \cdot \rho_{2F}(\vec{\epsilon}, \vec{M}, \vec{r}) d^3 \vec{\epsilon} d^3 \vec{M} \right) d^3 \vec{r} &= \int_{\vec{r} \in \Omega} \left[\frac{1}{2} \langle \vec{\epsilon} \wedge \vec{F} \rangle_2 Z \rho_1 \right] d^3 \vec{r} \\ &= \int_{\vec{r} \in \Omega} \mathbf{e} : \left[\frac{1}{2} \langle \vec{\epsilon} \otimes \vec{F} \rangle_2 Z \rho_1 \right] \\ &= \int_{\vec{r} \in \Omega} \mathbf{e} : \vec{G} d^3 \vec{r}. \end{aligned} \quad (\text{E2})$$

Now, if we ignore the errors of $\mathcal{O}(\|\vec{\epsilon} \cdot \nabla\|^3)$, (14) then reads

$$\int_{\vec{r} \in \partial \tilde{\Omega}} \vec{r} \wedge \vec{G} \cdot d\vec{A}(\vec{r}) = \int_{\vec{r} \in \Omega} (\nabla \cdot \vec{C} + \mathbf{e} : \vec{G}) d^3 \vec{r}$$

for arbitrary Ω . This equation together with (2) leads to (24). (*End of proof*)

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